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RESEARCH ARTICLE

Optimality Criteria Optimization of Truss Structures Under Multiple Frequency Constraints by the Linear Approximation Resizing Rule

Optimización mediante el criterio de optimalidad de estructuras de barras bajo restricciones de múltiples frecuencias por la aproximación de la regla lineal de redimensionado

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ABSTRACT

The optimization of structures requires an efficient method to minimize weight, while satisfying multiple types of constraints. This approach generalizes the optimality criteria for the specific type of constraints in the frequency. Equations of motion for truss structures are considered to obtain the derivatives of the constraints required by the optimality criterion. Exponential and linear resizing optimization rules for the design variables are described. In the first, the optimized areas are compared with the analytical solution for a continuous rod. As a second example the optimized frequencies, weights and areas obtained by the linear resizing rule are compared to reference values. Both examples demonstrate the validity and effectiveness of the optimality criteria approach for the frequency constraints in truss structures.

KEYWORDS: optimization; optimality criterion; structural design; linear approximation.

RESUMEN

La optimización de estructuras requiere un método eficiente para minimizar el peso, a la vez que satisface múltiples tipos de restricciones. El presente enfoque consiste en generalizar los criterios de optimalidad para el tipo específico de restricciones en la frecuencia. Se tienen en cuenta las ecuaciones de movimiento de las estructuras de celosía para obtener las derivadas de las restricciones requeridas por el criterio de optimalidad. Se describen las reglas de optimización de redimensionamiento exponencial y lineal para las variables de diseño. En el primer ejemplo, las áreas optimizadas se comparan con la solución analítica como una varilla continua. En el segundo ejemplo, las frecuencias, pesos y áreas optimizados obtenidos por la regla de redimensionamiento lineal se comparan con las referencias. Ambos ejemplos demuestran la validez y eficacia del enfoque de criterios de optimalidad para las restricciones de frecuencia en estructuras de armaduras.

PALABRAS CLAVE: optimización; criterio de optimalidad; diseño estructural; aproximación lineal.

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I. INTRODUCTION

Research on practical applications of the Optimality Criteria in truss structures under frequency restrictions by the linear approximation resizing rule has emerged as a critical area of inquiry due to its relevance in enhancing structural performance while ensuring dynamic stability [1], [2]. The evolution of this field spans from early topology optimization methods focusing on static criteria to advanced multi-criteria approaches incorporating natural frequency constraints and dynamic responses [3], [4]. Practical significance is underscored by the widespread use of lattice and truss structures in aerospace, civil, and mechanical engineering, where vibration control is essential to prevent resonance-induced failures [5], [6]. Studies report that optimized lattice structures can achieve significant weight reduction while increasing fundamental frequencies, directly impacting safety and efficiency [7], [8].

The specific problem addressed involves optimizing truss structures to satisfy frequency constraints [9], [10]. Despite advances, a notable knowledge gap persists in effectively integrating optimality criteria methods with linear approximation resizing rules under frequency constraints, especially in practical engineering applications [3], [11], [12]. Controversies arise regarding the best optimization algorithms; metaheuristic versus gradient-based and the handling of multiple frequency constraints and mode switching [10], [13], [14]. Failure to resolve these issues leads to suboptimal designs prone to dynamic instabilities and increased computational costs [15], [16].

Topology optimization defines material distribution, frequency constraints ensure dynamic performance, and linear approximation resizing facilitates efficient iterative updates. This framework guides the systematic review to evaluate how these concepts coalesce in practical truss structure design, grounded in structural dynamics and optimization theory [17], [18].

It is often important for structures subjected to dynamic loading that some frequencies are higher by some prescribed margin than the predominant frequencies. Furthermore, by using the well-developed optimality criterion approaches for design and efficient procedures for the implicit requirement of natural frequency analysis, the computational effort can be limited to within the same order of magnitude as that required for static loads analysis. This paper describes an effective optimality criterion computer design approach for member size selection to fulfill frequencies for truss structures. Optimality criteria methods use a relation that is based on the condition that a design is expected to meet at its optimum when the gradient for the frequency constraint is equal to the gradient of the objective function. Namely the weighted sum of the Lagrangian energy densities corresponding to multiple frequency constraints should be equal to unity in all the elements.

There are several works about the optimality criteria method in truss structures. For example, in [19] there are shown some optimality criterion algorithms for different constraints; however, such algorithms depend on nine control parameters, unlike the alone step-size parameter η used in the linear approximation. In [20], the linear approximation of the optimality criterion is applied, but only with stress and displacement constraints under static loading conditions.

The objective here is to develop an optimality criterion method for multiple constraints on the frequencies; that method should quickly reduce the weight of the starting design and converge to the optimal design variables. [21] and [22] have extended the method to multiple frequency constraints. This paper addresses the minimization of the structural mass in truss structures subject to frequency constraints under free vibration conditions. It differs from the works of [21], [22] and [23] in that the design scaling follows a different rule, so the use of the weighting parameter has been avoided. Just the step-size parameter is used.

II. METHODOLOGY

2.1. FREQUENCY ANALYSIS

The natural frequency analysis for a free undamped vibration of a discretized structure consists of finding a solution to the homogeneous set of a n-th order matrix represented by

$$[K]\psi_j - \omega_j^2[M + \overline{M}]\psi_j = 0 \qquad (1)$$
$$j = 1, 2, \ldots, n$$

where $[K]$, $[M]$, and $[\overline{M}]$ are the total stiffness, mass, and nonstructural mass matrices; ω_j is the circular frequency associated with the j-th vibration mode ψ_j. Multiplying Equation (1) by ψ_j^T gives

$$\psi_j^T[K]\psi_j - \omega_j^2\psi_j^T[M + \overline{M}]\psi_j = 0 \qquad (2)$$

Thus, solving for the circular frequency ω_j^2 we get

$$\omega_j^2 = \frac{(\psi_j^T[K]\psi_j)}{(\psi_j^T[M+\overline{M}]\psi_j)} \quad (3)$$

which is the Rayleigh quotient. The gradient of the circular frequency with respect to the design variable A_i is obtained by differentiating Equation (2); this gives

$$\frac{\partial\omega_j^2}{\partial A_i} = \frac{1}{A_i}\frac{(\psi_{j_i}^T[k]_i\psi_{j_i} - \omega_j^2\psi_{j_i}^T[m]_i\psi_{j_i})}{(\psi_j^T[M+\overline{M}]\psi_j)} \quad (4)$$

where ψ_{j_i} is the component of the vibration mode associated with the i-th element A_i, $[k]_i$, and $[m]_i$ are the stiffness and mass matrices of the i-th element. In deriving Equation (4) it was assumed that the stiffness and mass matrices are linear functions of the design variable A_i. According to [24], the stiffness and the consistent mass matrices of a bar are, respectively.

$$[K] = \frac{AE}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \quad (5)$$

$$[M]_c = \frac{\rho AL}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \quad (6)$$

In this manner, the derivatives of Equation (4) are valid for finite elements whose stiffness and mass matrices are linear functions of the area, such as demonstrated in Equations (5) and (6) for bars. Normalizing with respect to the mass matrix $[M+\overline{M}]$, Equation (4) is written as

$$\frac{\partial\omega_j^2}{\partial A_i} = \frac{1}{A_i}\{\tilde{\psi}_j\}_i^T[k]_i\{\tilde{\psi}_j\}_i - \omega_j^2\{\tilde{\psi}_j\}_i^T[m]_i\{\tilde{\psi}_j\}_i \quad (7)$$

where

$$\{\tilde{\psi}_j\}_i = \frac{\{\psi_j\}}{\left[\{\psi_j\}^T[M+\overline{M}]\{\psi_j\}\right]^{\frac{1}{2}}} \quad (8)$$

From now on, Equation (8) is being used in the forthcoming optimality criterion.

2.2. OPTIMIZATION PROCEDURE

The optimization problem is defined as minimizing the structural weight.

$$W(x_i) = \rho_i l_i x_i \quad i = 1, 2, \ldots, n \quad (9)$$

subject to m constraints

$$g_j(x_i) = \omega_i^2 - \overline{\omega}_j^2 = 0 \quad j = 1, 2, \ldots, k \quad (10)$$

$$g_j(x_i) = \omega_i^2 - \overline{\omega}_j^2 \leq 0 \quad j = k+1, k+2, \ldots, m \quad (11)$$

where ρ_i is the mass density, x_i is the design variable, and l_i is the length of the i element. In Equations (10) and (11), ω_j and $\overline{\omega}_j$ are the actual and desired values of the frequency constraints. In addition, minimum limits are prescribed on the design variables $x_i > x_i^l$.

Using Equations (9), (10) and (11), the Lagrangian function, L, as

$$L(x_i, \lambda) = \rho_i l_i x_i - \lambda_j(\omega_j^2 - \overline{\omega}_j^2) \quad i = 1, 2, \ldots, n \quad j = 1, 2, \ldots, m \quad (12)$$

where λ_j's are the Lagrange multipliers. Differentiating this equation with respect to the design variables and setting the resulting equations to zero, the optimality criterion leads to

$$\frac{\partial W(x_i)}{\partial A_i} - \sum_{j=1}^{m}\lambda_j\frac{\partial\omega_j^2}{\partial A_i} = 0 \quad (13)$$

i.e.

$$\rho_i l_i - \sum_{j=1}^{m}\lambda_j\frac{\partial\omega_j^2}{\partial A_i} = 0 \quad (14)$$

Thus, Lagrangian energy density is expressed as

$$\sum_{j=1}^{m}\frac{\lambda_j}{\rho_i l_i}\frac{\partial\omega_j^2}{\partial A_i} = 1 \quad (15)$$

Substituting the expression of the frequency derivative given by Equation (7) in Equation (15) yields

$$e_{ij} = \frac{\lambda_j\left[\{\tilde{\psi}_j\}_i^T[k]_i\{\tilde{\psi}_j\}_i - \omega_j^2\{\tilde{\psi}_j\}_i^T[m]_i\{\tilde{\psi}_j\}_i\right]}{A_i\rho_i l_i} = 1 \quad (16)$$

This represents the ratio of the gradient for the frequency constraint to the gradient of the objective function.

Now it is possible to write recursive relations to determine the Lagrange multipliers and modify the design variables. In both cases, these recursive relations are written in an exponential or linearized form. In the works of [21], [22], and [23] the exponential recursive relations are used, so that the design variables are modified by multiplying them by a quantity which is equal to unity at the optimum. The exponential rule is given by

$$x_i^{\text{new}} = x_i \left(\frac{1}{x_i^2 f_i} \sum_{j=1}^{n_g} \lambda_j c_{ij} \right)^{1/\eta} \qquad i = 1, 2, \ldots, n \tag{17}$$

where

$$f_i = \frac{\partial W(x_i)}{\partial x_i}, \qquad c_{ij} = -x_i^2 \frac{\partial g_j}{\partial x_i} \qquad i = 1, 2, \ldots, n \tag{18}$$

In the present work, the linear recursive relation is used for the estimation of the design variables, where they are modified by adding a quantity, Δx_i, which is equal to zero at the optimum. A linearized form of the Equation (17), obtained by binomial expansion, is

$$x_i^{\text{new}} = x_i + \Delta x_i \qquad i = 1, 2, \ldots, n \tag{19}$$

where

$$\Delta x_i = \frac{1}{\eta} \left(\frac{1}{x_i^2 f_i} \sum_{j=1}^{m} \lambda_j c_{ij} - 1 \right) x_i \qquad i = 1, 2, \ldots, n \tag{20}$$

and

$$\sum_{j=1}^{m} \sum_{i=1}^{n} \frac{c_{il} c_{ij}}{x_i^3 f_i} \lambda_j = \sum_{i=1}^{n} \frac{c_{il}}{x_i} \eta g_l(x_i) \qquad i = 1, 2, \ldots, m \tag{21}$$

The term η is a step size parameter. The old value for x_i is used to produce a new estimate. Note that the linear recursive relation for the Lagrange multipliers, Equation (21), approximates a set of linear equations that can be used to determine the Lagrange multipliers λ_j. Nonetheless, it is possible to investigate the convergence of these linear relations and compare them with the exponential relation results presented by the references.

III. RESULTS

3.1. ROD WITH A TIP MASS

Figure 1 shows the mechanical system analyzed herein presented by [25] and [23]. It is a continuous one-dimensional rod with a tip mass M_c = 0.1 lb s²/in. The rod is discretized in ten bar elements with equal length l_i = 9 in i = 1, …, 10. The fundamental axial frequency of the system is given a specified value, $\omega_1 \geq 2576.1$ rad/s = 410 Hz, and the sectional areas of the bars, A_i i = 1, …, 10, are the design variables required for which the total mass is a minimum. The rod is built from the linearly elastic material featuring modulus of elasticity E = 10.3 psi and density $\rho = 2.59\times10^{-4}$ lb/in³. The bar has ten degrees of freedom for joint translation in the horizontal direction, $u(y, t)$. According to [25], the analytical solution of the optimization problem constrained by the fundamental frequency leads to

$$A(x) = \frac{m(L)\cosh^2(\beta_1 L)}{\rho \cosh^2(\beta_1 y)} \tag{22}$$

where

$$m(L) = \beta_1 M_c \tanh(\beta_1 L) \tag{23}$$

and

$$\beta_1 = \omega\sqrt{(\rho/E)} \tag{24}$$

β_1 represents the frequency parameter associated with the fundamental mode.

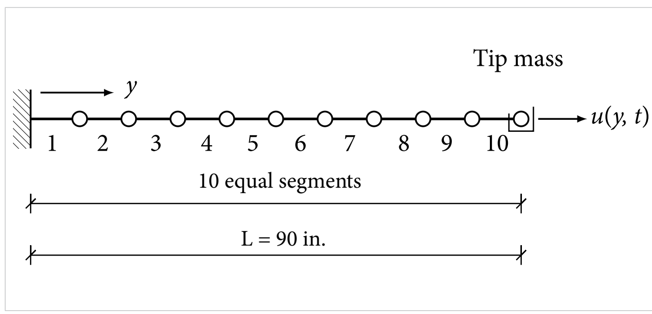

Figure 1. Ten bars structure.

Figure 2 shows the optimized cross-sectional areas obtained by both the analytical solution [25] and the here obtained optimization scheme for the discretized model. There is a difference that increases in the segments located at the left and right endpoints. In the left endpoint the optimization scheme underestimates the areas of the analytical function; in the right endpoint it overestimates these areas. A bigger match between both methods is observed in the range ($30 \leq y \leq 50$) in.

Figure 3 a) shows the optimization history for the first 10 iterations of the fundamental frequency here obtained. The starting values in all the areas of the bars are 9.3 in² for both the work of [23] and the present work. Figure 3 b) shows the weight iteration history. The linear optimization scheme exhibits a quick reduction of the starting weight and a quite fast convergence to the optimum values. The weight of the optimized structure by the analytical scheme is 80.44 lbs. The work of [23] reports a weight of 81.2 lbs. The weight of the optimized rod here obtained is 80.96 lbs., a lighter weight structure closer to the analytical weight value, despite the differences in the values of the area for the leftmost and the rightmost extremes shown in Figure 2.

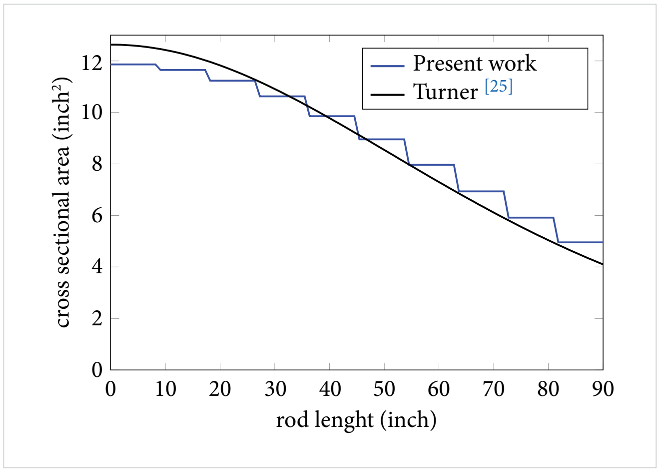

Figure 2. Optimal area distributions in the rod.

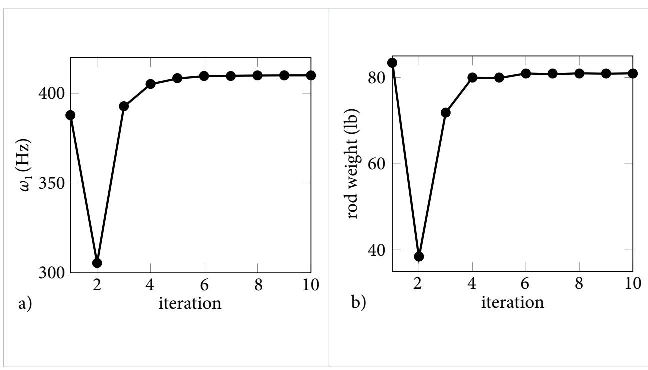

Figure 3. Iteration history: a) frequency history for ω_1 and b) rod weight history.

3.2. TEN BAR TRUSS

Figure 4 shows the ten-bar truss analyzed herein presented by [21]. It is a two-dimensional truss structure with tip masses $M_c = 2.588$ lb s^2/inch at the nodes 1, ..., 4. The structure is discretized in 10 bar elements from the linear elastic material featuring modulus of elasticity $E = 1{\times}10^7$ psi and density $\rho = 0.1$ lb/in^3. The axial frequencies of the system are given several specified values, and the sectional areas of the bars, A_i $i = 1, \ldots, 10$, are the design variables required for which the total mass is a minimum. There is a lower limit $x_l = 0.1$ for the design variables.

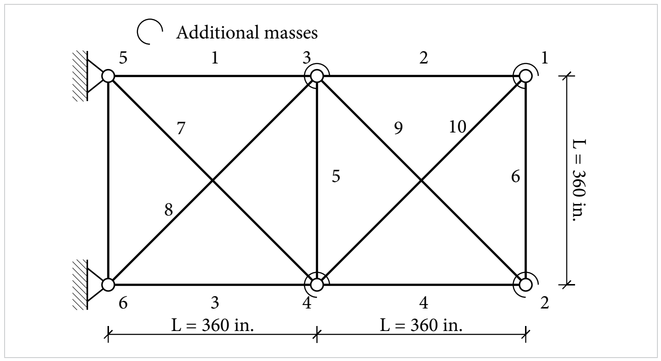

Figure 4. Ten-bar truss structure.

The optimization scheme utilizes initial values of 9.5318 in^2 across all bar areas for both, reference [21] and the present work. The second frequency must be 10, 15, 20, 25, 27.08, and 30 Hz across six distinct scenarios. Table 1 presents the optimal design frequencies and weights under specified constraints. The lines "%" show the absolute percentage difference between the reference values [21] and those obtained here. Only in the initial design are equal weights of 4000 lbs. for both, reference [21] and present work. A lighter structure results from the linear approximation rule in other restrictions. The major difference between both occurs at the restriction $\omega_2 = 30$ Hz, where the present work attains a 26.15% lighter weight than the reported by the reference [21].

TABLE 1
TEN-BAR TRUSS SECOND FREQUENCY VALUES (HZ) OBTAINED BY [21] AND PRESENT WORK (P.W.)

ω^2	Ref.	Frequency number								Weight (lbs)
		1	2	3	4	5	6	7	8	
Initial design	[21]	8.96	27.08	27.45	51.25	58	64.73	66.87	80.85	4000
	p.w.	9.177	27.31	29.78	53.8	61.05	68.34	69.93	82.09	4000
	%	2.42	0.85	8.49	4.98	5.26	5.58	4.58	1.53	0
10	[21]	3.26	10	10.19	16.01	18.08	22.96	25.21	27.25	304.5
	p.w.	3.179	9.992	10.16	13.36	13.92	19.33	25.13	26.28	269.45
	%	2.5	0.08	0.3	18.17	25.79	17.15	0.32	3.68	12.28
15	[21]	4.92	15.0	15.07	15.3	22.21	24.28	39.49	41.64	637.0
	p.w.	4.762	14.98	15.19	18.27	20.35	28.27	38.08	39.59	616.05
	%	3.27	0.13	0.79	17.52	8.76	15.28	3.65	4.97	3.36
20	[21]	6.64	20.0	20.13	21.51	30.39	32.81	52.53	55.89	1251.5
	p.w.	6.386	19.93	20.19	23.91	27.22	36.98	50.38	52.46	1158.4
	%	3.9	0.35	0.3	10.58	11.08	11.95	4.21	6.3	7.71
25	[21]	8.4	25	25.0	27.66	39.34	41.32	65.35	70.49	2243.8
	p.w.	8.095	24.91	25.23	28.93	34.73	44.81	62.02	65.08	1970.3
	%	3.7	0.36	0.92	4.48	12.44	8.15	5.23	8.07	12.87
27.8	[21]	9.17	27.08	27.11	30.96	43.96	45.54	70.4	76.84	2865.9
	p.w.	8.825	27.06	27.37	31.57	37.46	48.55	67.64	70.79	2424
	%	3.84	0.07	0.95	1.96	16.09	6.4	4.01	8.19	16.71
30	[21]	10.34	30.0	30.13	35.4	50.73	51.39	78.9	87.71	4143.9
	p.w.	9.867	29.98	30.34	34.76	41.68	53.28	74.42	78.06	3189.2
	%	4.68	0.07	0.69	1.83	19.34	3.64	5.85	12.28	26.15

The optimized area values are shown in Table 2. Despite the differences in the optimum design values, both results fulfill the frequency restrictions. There are no optimum variables at the lower limit x^l for this restriction.

TABLE 2
Ten-Bar Truss Area Values (inch²) Obtained by [21] and Present Work (p.w.)

ω^2	Ref.	Optimum design variables									
		1	2	3	4	5	6	7	8	9	10
10	[21]	0. 91	0.821	0.91	0.821	0.768	0.57	0.712	0.712	0.581	0.581
	p.w.	0.903	0.795	0.903	0.795	0.457	0.23	0.781	0.781	0.362	0.362
	%	0.77	3.22	0.77	3.22	50.6	82.35	9.25	9.25	46.81	46.81
15	[21]	2.313	2.154	3.313	2.154	0.602	0.353	1.723	1.723	1.037	1.036
	p.w.	2.23	1.888	2.23	1.888	1.098	0.534	1.785	1.785	0.793	0.793
	%	3.67	13.14	38.4	13.14	58.21	40.85	3.54	3.54	26.82	26.86
20	[21]	4.435	4.14	4.435	4.14	1.223	0.76	3.413	3.413	2.114	2.114
	p.w.	4.228	3.488	4.228	3.488	1.998	0.934	3.425	3.425	1.476	1.476
	%	4.78	17.15	4.78	17.15	48.71	20.61	0.35	0.35	35.34	35.34
25	[21]	7.699	7.224	7.697	7.223	2.195	1.382	6.211	6.211	4.017	4.003
	p.w.	7.038	5.708	7.038	5.708	3.07	1.394	6.11	6.11	2.637	2.637
	%	9.01	23.51	9	23.5	33.19	0.87	1.64	1.64	41.78	41.3
27.8	[21]	9.598	9.979	9.598	8.979	2.905	1.85	7.898	7.898	5.431	5.431
	p.w.	8.985	7.049	.985	7.049	3.896	1.703	7.428	7.428	3.057	3.057
	%	6.66	34.33	6.66	23.94	29.13	8.3	6.16	6.16	55.98	55.98
30	[21]	13.72	12.86	13.72	12.86	3.907	2.774	11.07	11.07	8.46	8.46
	p.w.	11.91	9.047	11.91	9.047	5.069	2.124	9.989	9.989	3.95	3.95
	%	14.19	34.78	14.19	34.78	25.87	26.65	10.2	10.2	71.86	71.86

The design of the ten-bar truss has now been developed, considering arrays of constraints pertaining to various frequency considerations that must be adhered to ensure optimal structural performance. The minimization of the structural weight is executed in such a manner that the fundamental frequency, along with the second and third frequencies, is subjected to specific constraints that ensure their values remain within predetermined limits.

The first set is with ω_1 = 7.0 Hz, the second with ω_1=10.0 Hz, the third with ω_1 = 7.0 Hz and ω_2 ≥15 Hz, the fourth with ω_1 = 10.0 Hz and ω_2 ≥ 15.0 Hz, the fifth with ω_1 = 7.0 Hz, ω_2 ≥ 15.0 Hz and ω_3 ≥ 20.0 Hz and, finally, all inequality constraints, ω_1 ≥ 3.5 Hz, ω_2 ≥ 10.0 Hz and ω_3 ≥ 14.0 Hz were considered. Table 3 presents the results obtained under these constraint conditions. It is observed that the present work results in a heavier weight optimized structure than the reference only in the first set; it is just a 0.14% increase in weight. For all the other sets the opposite happens; the linear approximation here used attains lighter weight optimized structures. The major difference occurs in the sixth set; it is a 13.43% difference in weights.

TABLE 3
Ten-Bar Truss Second Frequency Values (Hz) Obtained by [21] and Present Work (p.w.)

Set	Ref.	Frequency number								Weight (lbs)
		1	2	3	4	5	6	7	8	
1	[21]	7.0	10.96	16.27	18.21	27.39	29.55	47.92	50.34	1137.3
	p.w.	7	11.29	16.6	19.47	28.33	31.13	47.52	50.84	1138.9
	%	0	2.97	2.01	6.7	3.37	5.27	0.84	0.99	0.14
2	[21]	10	13.73	22.29	25.19	38.04	42.21	65.79	69.93	2614.0
	p.w.	10	14.38	23.04	25.64	37.83	39.89	65.8	69.17	2526.3
	%	0	4.64	3.31	1.77	0.55	5.67	0.02	1.09	3.41
3	[21]	7.0	15.58	16.93	18.75	29.13	30.3	46.93	49.67	1172.6
	p.w.	7	15	16.76	18.48	27.87	28.79	47.96	50.17	1158.8
	%	0	3.79	1.01	1.45	4.4	5.08	2.17	1	1.18
4	[21]	10	19.16	24.52	27.16	38.71	40.53	67.66	71.38	2736.3
	p.w.	10	15	23.29	27.23	39.45	43.15	64.95	70.06	2557.3
	%	0	24.57	5.14	0.26	1.9	6.22	4.08	1.86	6.76
5	[21]	7.0	15.61	20.17	20.77	28.76	29.76	53.88	56.03	1308.4
	p.w.	7	15.0	20.0	20.22	29.63	34.3	46.75	52.31	1243.2
	%	0	3.99	0.85	2.7	2.97	14.07	14.28	6.9	5.05
6	[21]	4.4	12.14	14.0	17.89	19.58	22.96	34.01	35.72	489.17
	p.w.	3.5	10.0	14.0	16.02	17.26	23.77	27.91	34.13	427.8
	%	22.65	19.57	0	11.02	12.5	3.46	19.86	4.55	13.43

Table 4 consists of all the design variables at the optimum. The variables A_5 and A_6 became passive after eight iterations for the first set. The variables A_5 and A_6 became passive after the third and ninth iterations respectively for the second set.

TABLE 4
Ten-Bar Truss Frequency Values (Hz) Obtained by [21] and Present Work (p.w.)

Set	Ref.	Optimum design variables									
		1	2	3	4	5	6	7	8	9	10
1	[21]	6.045	1.969	6.045	1.969	0.1	0.1	3.206	3.206	2.226	2.226
	p.w.	5.228	2.212	5.228	2.212	0.1	0.1	3.418	3.418	2.434	2.434
	%	14.5	11.66	14.5	11.66	0	0	6.44	6.44	8.92	8.92
2	[21]	13.965	4.437	13.965	4.437	0.1	0.1	7.579	7.579	5.009	5.009
	p.w.	12.176	4.503	12.176	4.503	0.1	0.1	7.93	7.93	5.014	5.014
	%	13.78	1.47	13.78	1.47	0	0	4.52	4.52	0.1	0.1
3	[21]	5.511	1.937	5.511	1.937	0.207	0.414	3.616	3.616	2.414	2.414
	p.w.	5.251	2.222	5.251	2.222	0.299	0.298	3.438	3.438	2.446	2.446
	%	4.84	13.76	4.84	13.76	37.07	32.89	5.08	5.08	1.32	1.32
4	[21]	13.147	5.683	13.147	5.683	0.488	0.517	9.093	9.093	4.11	4.11
	p.w.	12.178	4.541	12.178	4.541	0.326	0.111	8.079	8.079	5.058	5.058
	%	7.69	22.25	7.69	22.25	39.75	128.42	11.75	11.75	20.59	20.59
5	[21]	5.672	3.823	5.672	3.823	0.646	0.321	4.191	4.191	1.604	1.604
	p.w.	6.039	0.819	5.46	2.875	0.871	0.584	3.417	3.83	3.247	2.126
	%	6.26	134.18	3.83	28.32	29.7	57.75	20.4	8.92	67.09	27.6
6	[21]	2.306	1.304	2.306	1.304	0.639	0.557	1.029	1.029	0.8	0.8
	p.w.	0.749	0.514	0.98	1.571	1.077	0.408	2.103	1.17	0.687	0.692
	%	104.91	85.39	80.08	18.78	51.27	30.73	68.21	12.98	15.22	14.54

The linear approximation scheme here presented attains lighter weight structures in almost all cases of restrictions on the first, second and third frequencies; it was demonstrated how the linear approximation here used can effectively optimize the truss structures here studied. Optimization of the structure to attain specific frequencies can be achieved by using the linear approximation, which is a function of the stiffness, the structural and the non-structural masses.

V. CONCLUSIONS

This study has demonstrated that the application of optimality criteria, particularly through the linear resizing rule, provides an effective strategy for addressing structural optimization problems under multiple frequency constraints in discretized truss systems. Across the examples analyzed, the proposed formulation not only achieved significant weight reductions but also ensured compliance with the imposed dynamic conditions, while maintaining consistency with reference results reported in the literature.

In the rod with a tip mass example, the optimized cross-sectional areas showed an elevated level of agreement with the continuous analytical solution, confirming the robustness of the method when applied to discretized models. Although slight deviations were observed at the ends of the bar, the linear approximation produced a final weight closer to the analytical value, which highlights its ability to converge rapidly toward optimal designs without compromising accuracy.

Similarly, in the ten-bar truss case, the method once again proved effective by yielding solutions that, in most scenarios, resulted in lighter structures than those obtained in previous studies, while still fulfilling all frequency requirements. It is worth noting that weight reductions were particularly significant under more demanding constraint conditions, which underscores the advantages of the linear approximation over traditional schemes based on exponential rules or multiple control parameters.

A noteworthy aspect of this approach is that the linear resizing rule relies solely on a single step-size parameter, thereby avoiding the need for additional weighting factors. This not only simplifies its implementation but also enhances its adaptability to different structural types and design scenarios, broadening its potential for practical applications.

Overall, the results confirm that the proposed method achieves a favorable balance between computational efficiency, operational simplicity, and reliability of optimized designs. These attributes position it as a solid alternative for structural optimization under dynamic constraints, in comparison with other approaches that, although effective, typically demand greater parametrization and higher computational effort.

Finally, it is emphasized that, while the case studies confirm the validity of the method, future research may extend its application to problems involving nonlinear dynamic loads, damping effects, and three-dimensional structures. Advancing in these directions would further consolidate the linear approximation of optimality criteria as a reliable tool for the advanced design of lightweight and efficient structures subject to dynamic performance requirements.

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