A note on Dulac functions for a sum of vector fields

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Abstract

In this paper we give sufficient conditions for constructing Dulac functions of a sum of planar vector fields. We give some applications and examples in order to illustrate the results.

Key words: Bendixson-Dulac criterion, Dulac functions, limit cycles.

1 Introduction

Let \( F(x_1, x_2) = (f_1(x_1, x_2), f_2(x_1, x_2)) \) be a planar vector field, with components \( f_1, f_2 \) \( C^1 \)-functions defined on an open set \( \Omega \) which by convenience we consider simply connected. We consider

\[
\begin{aligned}
\dot{x}_1 &= f_1(x_1, x_2), \\
\dot{x}_2 &= f_2(x_1, x_2), \quad (x_1, x_2) \in \Omega,
\end{aligned}
\]

in short write,

\[
\dot{x} = F(x), \quad x = (x_1, x_2) \in \Omega.
\]

It is well known that many questions in applications of systems (1) are related with periodic orbits, and of particular interest are the existence or nonexistence of such orbits, number of them, among others; see [6].

Along the work we will use the Lebesgue measure, now define the sets

\[
\mathcal{F}_\Omega^\pm := \{ f \in C^0(\Omega, \mathbb{R}^\pm \cup \{0\}) : \text{vanishes at most on a measure zero set}\},
\]

where as usual \( C^0 \) denotes the set of continuous functions and consider

\[
\mathcal{F}_\Omega := \mathcal{F}_\Omega^- \cup \mathcal{F}_\Omega^+.
\]
For the simply connected region $\Omega$ and $F$, we introduce the sets
\[ D_{\Omega}^{\pm}(F) = \{ h \in C^1(\Omega, \mathbb{R}) : div(hF) := \frac{\partial(hf_1)}{\partial x_1} + \frac{\partial(hf_2)}{\partial x_2} \in F_{\Omega}^{\pm} \}. \]

We recall that a limit cycle of the differential equation (1) is a periodic orbit of this equation which is isolated in the set of all periodic orbits of the system (1).

A classical result to rule out existence of periodic orbits or limit cycles is the Bendixson-Dulac criterion, and for convenience we recall this result (see [3], p. 137).

**Theorem 1.1. (Bendixson-Dulac criterion)** Let $f_1(x_1, x_2)$, $f_2(x_1, x_2)$ and $h(x_1, x_2)$ be functions $C^1$ in a simply connected domain $\Omega \subset \mathbb{R}^2$ such that $\frac{\partial(f_1h)}{\partial x_1} + \frac{\partial(f_2h)}{\partial x_2} \in F_{\Omega}$. Then the system (1) does not have periodic orbits in $\Omega$.

A function as in the theorem is called a *Dulac function*. Note that a Dulac function of (1) is an element in the set
\[ D_{\Omega}(F) := D_{\Omega}^{+}(F) \cup D_{\Omega}^{-}(F). \]

It is well known that Bendixson-Dulac criterion is a very useful tool for investigation of limit cycles (see [1], [2], [5]) or also to discard existence of polycycles. Despite the relevance of Bendixson-Dulac’s criterion it suffers the drawback that there is no general algorithm for finding $h$ functions. In this note we study the problem of when the sum of two vector fields admits a Dulac function and consequently has no periodic orbits.

### 2 Results

The following gives information on Dulac functions for linear relationships of vector fields.

**Lemma 2.1.** Let $\Omega \subset \mathbb{R}^2$ be a simply connected set and $X_i := p_i(x_1, x_2) \frac{\partial}{\partial x_1} + q_i(x_1, x_2) \frac{\partial}{\partial x_2}$ for $i = 1, 2$, be two $C^1$ vectors fields defined on $\Omega$ with $X_1 \pm X_2$ not identically zero on any neighbourhood, the following holds:

a) If $h \in D_{\Omega}(X_i)$, then $h \in D_{\Omega}(X_1 + X_2)$ or $h \in D_{\Omega}(X_1 - X_2)$.

b) If $h \in D_{\Omega}^{+}(X_i)$, then $h \in D_{\Omega}^{+}(X_1 + X_2)$.

c) If $h \in D_{\Omega}^{-}(X_1)$, then $h \in D_{\Omega}^{-}(cX_1)$ for $c \in \mathbb{R}\setminus\{0\}$. 
Proof. We consider the case a). We have
\[ \text{div}(hX_1)\text{div}(hX_2) \in \mathcal{F}_\Omega^+ \text{ or } \text{div}(hX_1)\text{div}(hX_2) \in \mathcal{F}_\Omega^-; \]
in the first case \( h \in \mathcal{D}_\Omega(X_1 + X_2) \) and in the second case \( h \in \mathcal{D}_\Omega(X_1 - X_2). \)

Recall that a function \( f \in C^1(\Omega, \mathbb{R}) \) is an integrating factor of the vector field \( X \) if \( \text{div}(fX) = 0. \)

**Proposition 2.2.** Let \( \Omega \) be a simply connected open set and \( F_1, F_2 \in C^1(\Omega, \mathbb{R}^2) \) be vector fields. Assume that \( h_i \in \mathcal{D}_\Omega^\pm(F_i) \) for \( i = 1, 2 \) and \( h_i \) is an integrating factor of \( F_j \) for \( 1 \leq i \neq j \leq 2 \), then \( h_1 + h_2 \in \mathcal{D}_\Omega^\pm(F_1 + F_2) \), in particular, the sum \( F_1 + F_2 \) does not admit periodic orbits or limit cycles entirely contained in \( \Omega \).

**Proof.** A direct calculation gives
\[
\text{div}((h_1 + h_2)(F_1 + F_2)) = \text{div}(h_1F_1) + \text{div}(h_2F_1) + \text{div}(h_1F_2) + \text{div}(h_2F_2) = \text{div}(h_1F_1) + \text{div}(h_2F_2),
\]
therefore, \( h_1 + h_2 \in \mathcal{D}_\Omega(F_1 + F_2). \)

The following result in [4] gives conditions on which vector fields we can add to one given such that the resulting system admits a Dulac function:

**Proposition 2.3.** Assume that \( h \in \mathcal{D}_\Omega^\pm(F) \), then for every \( C^1 \) vector field \( G \) with \( (\pm)\text{div}(hG) \geq 0 \) we have \( h \in \mathcal{D}_\Omega(F + G) \), in particular, the sum \( F + G \) does not admit periodic orbits or limit cycles entirely contained in \( \Omega \).

We remark that proposition 2.3 can be used for to simplify the construction of Dulac functions of some systems, in effect, we write the vector field as the sum of two vector fields with some specific properties

**Example 2.4.** Consider the model for the interaction between two species given by
\[
\begin{align*}
\dot{x}_1 &= x_1x_2 - x_1 + \mu x_2 = f_1, \\
\dot{x}_2 &= x_2 - x_1x_2 + \epsilon x_1 = f_2,
\end{align*}
\]
we can write \((f_1, f_2) = (x_1x_2 - x_1, x_2 - x_1x_2) + (\mu x_2, \epsilon x_1): \) note that if \( h(x_1, x_2) = \frac{1}{x_1x_2} \) then \( \text{div}(h(x_1x_2 - x_1, x_2 - x_1x_2)) = 0 \) and \( \text{div}(h(\mu x_2, \epsilon x_1)) < 0, \) thus \( h \) is a Dulac function on \( \mathbb{R}_+^2 := \{(x_1, x_2) \in \mathbb{R}^2 : x_1 > 0, x_2 > 0\}. \)
Example 2.5. Consider the system

\[
\begin{align*}
\dot{x}_1 &= x_2 = f_1, \\
\dot{x}_2 &= -x_1 - x_2 + x_1^2 + x_2^2 = f_2,
\end{align*}
\]

note that if we assume a Dulac function \( h \) depending only on \( x_1 \) then \( \text{div}(h(0, -x_1 - x_2 + x_1^2)) = -h(x_1) \), so we need to find a function that does not change sign and such that \( \text{div}(h(x_2, x_2^2)) = 0 \), i.e., \( 0 = \frac{\partial h}{\partial x_1} x_2 + 2hx_2 \), but this equation admits the solution \( h(x_1) = \exp(-2x_1) \); by proposition 2.3 this is a Dulac function of the system.

Example 2.6. We consider the classic susceptible–exposed–infectious–susceptible (SEIS) epidemic model with constant population \((\eta)\)

\[
\begin{align*}
\dot{x}_0 &= \mu\eta - \beta x_0 x_2 - \mu x_0 + \delta x_2, \\
\dot{x}_1 &= \beta x_0 x_2 - (\epsilon + \mu)x_1, \\
\dot{x}_2 &= \epsilon x_1 - (\delta + \mu)x_2,
\end{align*}
\]

with positive parameters, again we consider

\[
\Sigma = \left\{ (x_0, x_1, x_2) \in \mathbb{R}_+^3 : x_0 \geq 0, x_1 \geq 0, x_2 \geq 0, x_0 + x_1 + x_2 = \eta \right\}.
\]

The system (3) is subject to the restriction \( x_0 + x_1 + x_2 = \eta \), and using \( x_0 = \eta - x_1 - x_2 \) in the model, we can eliminate \( x_0 \) from the equations. This substitution gives the simple model

\[
\begin{align*}
\dot{x}_1 &= \beta(\eta - x_1 - x_2)x_2 - (\epsilon + \mu)x_1 = f_1, \\
\dot{x}_2 &= \epsilon x_1 - (\delta + \mu)x_2 = f_2,
\end{align*}
\]

has \( h(x_1, x_2) = x_1^{-1} \) as a Dulac function, we consider a term of transmission between exposed and susceptible individuals \( G := (\kappa(\eta - x_1 - x_2)x_1, 0) \) and note that \( \text{div}(hG) = -\kappa < 0 \); by above proposition we get \( h \in \mathcal{D}_\Sigma(F + G) \).

Now we analyse other conditions such that a sum of vector fields admit Dulac functions.

Proposition 2.7. Assume that \( 0 \neq h \in \mathcal{D}_\Omega^+(F) \) and \( \Delta h \geq 0 \), i.e., an harmonic or subharmonic function, then we have \( \mathcal{D}_\Omega(F + \nabla h) \neq \emptyset \).

Proof. A direct calculation yields

\[
\text{div}(h\nabla h) = (\nabla h)^2 + \Delta h \geq 0,
\]

the result follows from proposition 2.3.
**Example 2.8.** Consider the system given in example 2.4, note that the Dulac function \( h(x_1, x_2) = \exp(-2x_1) \) obtained is a subharmonic function, i.e., \( \Delta h \geq 0 \), therefore if to original vector field we add the gradient \( \nabla h \), then the new vector field admits no periodic orbits.

**Proposition 2.9.** Let \( h \) be a Dulac function of \( F \) with \( \text{div}(hF) \geq 0 \), suppose that \( h \geq 0 \), \( h^{-1}(c) \) bounds a convex region for all \( c \in \mathbb{R}^+ \) and \( h(p) = 0 \) for some \( p \). Let \( G \) be a vector field with \( p \) critical point, and \( \text{div}(G) > 0 \), \( \forall x \neq p \), then \( \mathcal{D}_{\mathbb{R}^2}(F + G) \neq \emptyset \), in particular, the sum \( F + G \) does not admit periodic orbits or limit cycles entirely contained in \( \Omega \).

**Proof.** First by the convexity assumption the gradient \( \nabla h \) is outside of level sets \( h^{-1}(c) \) for \( c > 0 \) fixed.

As \( \text{div}(G) > 0 \), \( \forall x \neq p \), the flux of \( G \) is expanded, so that in each point of the level set \( h^{-1}(c) \) we have that the angle

\[
\theta := \angle(G, \nabla h) \in (0, \frac{\pi}{2}),
\]

therefore, \( G \cdot \nabla h = ||G||||\nabla h|| \cos(\theta) \geq 0 \) where, as usual \( \cdot \) denotes the scalar product, thus

\[
\text{div}(hG) = G \cdot \nabla h + h\text{div}(G) \geq 0,
\]

the result follows by proposition 2.3. \( \square \)

**Example 2.10.** The system

\[
\begin{align*}
\dot{x}_1 &= x_2 + x_1 x_2, \\
\dot{x}_2 &= -x_1 + x_1^2 x_2,
\end{align*}
\]

has \( h(x_1, x_2) = x_1^2 + x_2^2 \) as a Dulac function, we consider \( G := (x_1 x_2^2 + x_1^3 x_2, x_2 x_1^4) \) and note that \( \text{div}(G) = x_2^2 + 3x_1^2 x_2 + x_1^4 \geq 0 \); by above proposition we get \( \mathcal{D}_{\mathbb{R}^2}(F + G) \neq \emptyset \).

Recall that a (nonempty) open set \( \Omega_0 \) is compactly contained in an open set \( \Omega \subseteq \mathbb{R}^2 \), denoted \( \Omega_0 \subsetsubset \Omega \), if \( \overline{\Omega_0} \subset \Omega \) and \( \overline{\Omega_0} \) is compact. Note that if \( \Omega_0 \subsetsubset \Omega \), then there is an open set \( V \) such that \( \Omega_0 \subsetsubset V \subsetsubset \Omega \).

In [4], it was established that if \( \mathcal{D}_{\Omega_0}(F) \neq \emptyset \) for all \( \Omega_0 \subsetsubset \Omega \), then the system (2) admits no periodic orbits in \( \Omega \).
Proposition 2.11. Let $\Omega$ be a simply connected open set and $F, G \in C^1(\Omega, \mathbb{R}^2)$ be vector fields. Suppose there exists $h: \Omega \to \mathbb{R}$ never zero, such that $F \cdot \nabla h > 0$ and $G \cdot \nabla h \geq 0$, then $\mathcal{D}_{\Omega_0}(F + G) \neq \emptyset$, for all $\Omega_0 \subset \subset \Omega$, in particular, $F + G$ does not admit periodic orbits entirely contained in $\Omega$.

Proof. Take an open set $\Omega_0$ compactly contained in $\Omega$. We choose an open set $V$ such that $\Omega_0 \subset \subset V \subset \subset \Omega$. Let $n \in \mathbb{N}$ an odd number such that 

$$nF \cdot \nabla h > |h[\text{div}(F) + \text{div}(G)]|$$

on $\overline{V}$, which is possible by the compactness of $\overline{V}$. Since $n - 1$ is even, then $h^{n-1} > 0$ on $\Omega$ and we get

$$\text{div}(h^n(F + G)) = \text{div}(h^n F) + \text{div}(h^n G) = h^{n-1}(nF \cdot \nabla h + nG \cdot \nabla h + h[\text{div}(F) + \text{div}(G)])$$

therefore $\text{div}(h^n(F + G)) > 0$ on $\overline{V}$, in particular, $\text{div}(h^n(F + G)) > 0$ on $\Omega_0$ hence $h^n \in \mathcal{D}_{\Omega_0}(F + G)$, which proves the result. 

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